1 For $(x,y) \neq (0,0)$, we may write

$x = r\cos\theta$, $y = r\sin\theta$ for some $r > 0$, $\theta \in [0, 2\pi]$

$\left(r = \sqrt{x^2+y^2} \right)$

Then,

$$|f(x,y)| = |r\cos\theta(r\sin\theta + 1)\ln r^2|$$

$$\leq 2r(r+1)|\ln r|$$

$$\boxed{\leq} \; 4r|\ln r|$$

when $0 < r < 1$

Moreover, $\lim_{r\to 0^+} 4r\ln r = 4\lim_{r\to 0^+} \frac{\ln r}{\frac{1}{r}}$ $\left(\frac{\infty}{\infty}\right)$

(by L' Hôpital's rule) $= 4\lim_{r\to 0^+} \frac{\frac{1}{r}}{-\frac{1}{r^2}}$

$= 0$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0 \neq f(0,0)$

$\therefore$ f is not continuous at $(0,0)$ and the discontinuity is removable

(If we redefine $f(0,0) = 0$, then f is cont. at $(0,0)$)

2. $$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

3. Tree diagram :

$$\begin{tikzcd}
 & & r \\
 & x \arrow[ur] \arrow[dr] & \\
u \arrow[ur] \arrow[dr] & & \theta \\
 & & r \\
 & y \arrow[ur] \arrow[dr] & \\
 & & \theta
\end{tikzcd}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y} \cdot \sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)^2$$

$$+\left(-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta\right)^2$$

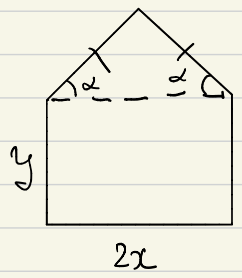
$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

4. The distance between the plane and the origin is $\left|\frac{D}{\sqrt{A^2+B^2+C^2}}\right|$

The point on the plane that is nearest to the origin $(0,0,0)$ is

$$\frac{(A, B, C)}{\sqrt{A^2+B^2+C^2}} \cdot \frac{-D}{\sqrt{A^2+B^2+C^2}} = \frac{-D}{A^2+B^2+C^2}(A,B,C)$$

5.



Let P be the perimeter

Then, $P = 2x + 2y + \frac{2x}{\cos\alpha}$ — ①

Area $A = 2xy + x^2 \tan\alpha$ — ②

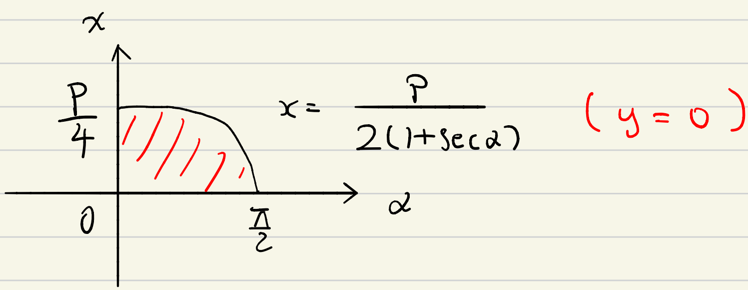
By ①, $$y = \frac{1}{2}\left[P - 2x - \frac{2x}{\cos\alpha}\right]$$

$$A(x,\alpha) = x\left(P - 2x - \frac{2x}{\cos\alpha}\right) + x^2\tan\alpha$$

We want to maximize $A(x,\alpha)$ subject to some constraints of x and α.

① $0 \leq x \leq \frac{P}{4}$, $0 \leq \alpha \leq \frac{\pi}{2}$

② $0 \leq P - 2x - \frac{2x}{\cos\alpha} \leq P$



Boundary case :

① $x=0$: $A = 0$

② $\alpha = 0$: $A = x(P-4x)$

$= -4x^2 + Px$

$= -4(x - \frac{P}{8})^2 + \frac{P^2}{16}$

$\therefore$ Area is maximized at $x = \frac{P}{8}$,

$A(0, \frac{P}{8}) = \frac{P^2}{16}$

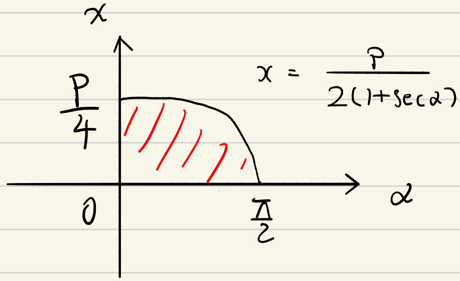
③ $x = \frac{P}{2(1+\sec\alpha)}$ or $(y=0)$

For triangles with fixed perimeter, the equilateral triangle has the maximum possible area.

$\therefore$ Area is maximized at $x = \frac{P}{6}$, $\alpha = \frac{\pi}{3}$

$A(\frac{P}{6}, \frac{\pi}{3}) = \frac{P^2}{36}\sqrt{3}$

Consider the interior of the domain below



$(y = 0)$

Set

$$\begin{cases} \dfrac{\partial A}{\partial x} = P - 4x - \dfrac{4x}{\cos\alpha} + 2x\tan\alpha = 0 \\ \dfrac{\partial A}{\partial \alpha} = -2x^2 \sec\alpha \tan\alpha + x^2 \sec^2\alpha = 0 \end{cases}$$

$$\frac{\partial A}{\partial \alpha} = 0 \quad \Rightarrow \quad 2\sin\alpha = 1$$

$$\Rightarrow \quad \alpha = \frac{\pi}{6}$$

$$\frac{\partial A}{\partial x}\left(x, \frac{\pi}{6}\right) = 0 \Rightarrow P - 4x - 4x\frac{2}{\sqrt{3}} + 2x\frac{1}{\sqrt{3}} = 0$$

$$\Rightarrow \quad P = x(2\sqrt{3} + 4)$$

$$\Rightarrow \quad x = \frac{P}{2}\left(\frac{1}{\sqrt{3}+2}\right)$$

$$= \frac{P}{2}(2 - \sqrt{3})$$

$$A\left(\frac{P}{2}(2-\sqrt{3}), \frac{\pi}{6}\right)$$

$$= \frac{P}{2}(2-\sqrt{3})\left(P - P(2-\sqrt{3}) - P(2-\sqrt{3})\cdot\frac{2}{\sqrt{3}}\right)$$

$$+ \frac{P^2}{4}(2-\sqrt{3})^2\frac{1}{\sqrt{3}}$$

$$= \frac{P^2}{2}(2-\sqrt{3}) - \frac{P^2}{2}(2-\sqrt{3})^2\left(1+\frac{2}{\sqrt{3}}\right)$$

$$+ \frac{P^2}{4}(2-\sqrt{3})^2\frac{1}{\sqrt{3}}$$

$$= \frac{P^2}{2}(2-\sqrt{3}) - \frac{P^2}{2}(2-\sqrt{3})^2 - \frac{3}{4}P^2(2-\sqrt{3})^2\frac{1}{\sqrt{3}}$$

$$= \frac{P^2}{4}(2-\sqrt{3})\left(2 - 2(2-\sqrt{3}) - \frac{3(2-\sqrt{3})}{\sqrt{3}}\right)$$

$$= \frac{P^2}{4}(2-\sqrt{3})(1) = \frac{P^2}{4}(2-\sqrt{3})$$

Among $A(0, \frac{P}{8}) = \frac{P^2}{16}$, $A(\frac{P}{6}, \frac{\pi}{3}) = \frac{P^2\sqrt{3}}{36}$

and $A\left(\frac{P}{2}(2-\sqrt{3}), \frac{\pi}{6}\right) = \frac{P^2}{4}(2-\sqrt{3})$

$A\left(\frac{P}{2}(2-\sqrt{3}), \frac{\pi}{6}\right)$ is the largest

$\therefore$ The maximum possible area is $\frac{P^2}{4}(2-\sqrt{3})$

6. Let $(a, b, \sqrt{a^2+b^2})$ be a point on the cone.

Normal vector of the tangent plane at

$$(a, b, \sqrt{a^2+b^2}) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \right)\Bigg|_{(x,y)=(a,b)}$$

$$= \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, -1 \right)$$

Equation of the tangent plane is

$$\frac{a}{\sqrt{a^2+b^2}}(x-a) + \frac{b}{\sqrt{a^2+b^2}}(y-b) - \left(z - \sqrt{a^2+b^2}\right) = 0$$

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y - z = 0$$

$\therefore$ $(0,0,0)$ is lying on the tangent plane.

7. Let $(a,b,c) \in \mathbb{R}^3$ so that $ab = 4$

It is a point lying on the vertical cylinder.

Normal vector of the tangent plane at (a,b,c) is $\nabla f \big|_{(a,b,c)}$, where

$f(x,y,z) = xy - 4$

Hence, $\nabla f \big|_{(a,b,c)} = (b, a, 0)$

Equation of the tangent plane is

$$b(x-a) + a(y-b) = 0$$

$$bx + ay = 2ab$$

Its distance from the origin is

$$\left| \frac{2ab}{\sqrt{a^2+b^2}} \right| = \frac{8}{\sqrt{a^2+b^2}}$$

A plane is tangent to the sphere $x^2+y^2+z^2 = 8$ iff it has distance $\sqrt{8}$ from the origin

$\therefore$ The tangent plane of the vertical cylinder at (a,b,c) is tangent to the sphere iff

$$a^2 + b^2 = 8$$

$$\Leftrightarrow \quad a^2 - 2ab + b^2 = 0 \quad (\because ab = 4)$$

$$\Leftrightarrow \quad (a-b)^2 = 0$$

$$\Leftrightarrow \quad a = b$$

i.e. $(a,b) = (2,2)$ or $(-2,-2)$

All possible planes are

$x+y = 4$ and $x+y = -4$

8. $$\frac{\partial f}{\partial x} = 2xy + z^2, \quad \frac{\partial f}{\partial y} = 2yz + x^2$$

$$\frac{\partial f}{\partial z} = 2xz + y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x, \quad \frac{\partial^2 f}{\partial z \partial x} = 2z$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x, \quad \frac{\partial^2 f}{\partial z \partial y} = 2y$$

$$\frac{\partial^2 f}{\partial x \partial z} = 2z, \quad \frac{\partial^2 f}{\partial y \partial z} = 2y$$

$$\therefore \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x},$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$

It is possible to find a function $f(x, y, z)$ with the given partial derivatives.

$$\frac{\partial f}{\partial x} = 2xy + z^2$$

$$f(x,y,z) = \int \frac{\partial f}{\partial x} \, dx + g(y,z)$$

$$= x^2 y + xz^2 + g(y,z) \quad \text{—} (\ast)$$

From $(\ast)$, $\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y}$

On the ~~other~~ hand, it is given that

$$\frac{\partial f}{\partial y} = 2yz + x^2$$

$$\therefore \quad \frac{\partial g}{\partial y} = 2yz$$

$$\Rightarrow \quad g(y,z) = y^2 z + h(z)$$

i.e. $$f(x,y,z) = x^2 y + xz^2 + y^2 z + h(z)$$

$$\frac{\partial f}{\partial z} = 2xz + y^2 + h'(z) = 2xz + y^2$$

computation

given

$\therefore \quad h(z) = C$. is a constant fcn.

$$\therefore \quad f(x,y,z) = x^2 y + xz^2 + y^2 z + C$$